Consideration of the Difference-Differential equation $(a+bx)ddz+(c+ex)\frac{dx\cdot dz}{x}+(f+gx)\frac{zdx^2}{xx}=0$*

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§1 First, one can reduce this equation to a simple differential form by putting $\log z = \int vdx$ such that

$$\frac{dz}{z}=vdx \quad \text{and} \quad \frac{ddz}{z}-\frac{dz^2}{zz}=dxdv$$

and hence

$$\frac{ddz}{z}=dxdv+vvdx^2.$$

For, having divided that equation by zdx, from this

$$(a+bx)dv+(a+bx)vvdx+(c+ex)\frac{vdx}{x}+(f+gx)\frac{dx}{xx}=0$$

will result; if its integral would be known, it would be $\log z=\int vdx$ for the propounded one.

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§2 Hence the term affected by the simple quantity v can be thrown out in two ways. For the one let us set $v = u + X$, while X denotes a function of x that is to be determined soon, and after the substitution it will be

$$\left.\begin{aligned} (a+bx)du + (a+bx)uudx + (c+ex)\frac{udx}{x} \quad &+ (f+gx)\frac{dx}{xx} \\ + 2(a+bx)uXdx \quad &+ (a+bx)dX \\ &+ (a+bx)XXdx \\ &+ (c+ex)\frac{Xdx}{x} \end{aligned}\right\} = 0.$$

Now set

$$X = \frac{-(c+ex)}{2x(a+bx)}$$

such that

$$dX = \frac{(ac+2bcx+bexx)dx}{2xx(a+bx)^2},$$

and

$$\left.\begin{aligned} (a+bx)du + (a+bx)uudx + (f+gx)\frac{dx}{xx} &+ \frac{(ac+2bcx+bexx)dx}{2xx(a+bx)} \\ &- \frac{(cc+2cex+eexx)dx}{4xx(a+bx)} \end{aligned}\right\} = 0$$

will result or in this way:

$$du + uudx + \frac{4(a+bx)(f+gx) + 2(ac+2bcx+bexx) - (c+ex)^2}{4xx(a+bx)^2}dx = 0.$$

§3 By the other method combined with the first put $v = Pu + X$, while P and X are functions of x, and after the substitutions one will obtain

$$\left.\begin{array}{rr}(a+bx)Pdu+(a+bx)PPuudx+(c+ex)\dfrac{Pudx}{x} & +\dfrac{(f+gx)dx}{xx}\\ +(a+bx)udP & +(a+bx)dX\\ +2(a+bx)PXudx & +(a+bx)XXdx\\ & +(c+ex)\dfrac{Xdx}{x}\end{array}\right\}=0,$$

whence it has to be

$$\frac{(c+ex)Pdx}{x}+(a+bx)dP+2(a+bx)PXdx=0.$$

But I introduced the two functions P and X here for the investigation to extend further; for, using this other method we usually put $v=Pu$ that $X=0$, in which case it will be

$$\frac{dP}{P}+\frac{(c+ex)dx}{x(a+bx)}=0 \quad \text{or} \quad \frac{dP}{P}+\frac{c}{a}\cdot\frac{dx}{x}+\frac{ae-bc}{a}\cdot\frac{dx}{a+bx}=0,$$

from which by integration one concludes:

$$Px^{\frac{c}{a}}(a+bx)^{\frac{ae-bc}{ab}}=C \quad \text{and} \quad P=Cx^{\frac{-c}{a}}(a+bx)^{\frac{c}{a}-\frac{e}{b}}$$

and our differential equation of first degree will be

$$Cx^{\frac{-c}{a}}(a+bx)^{\frac{c}{a}-\frac{e}{b}+1}du+CCx^{\frac{-2c}{a}}(a+bx)^{\frac{2c}{a}-\frac{2e}{b}+1}uudx+\frac{(f+gx)dx}{xx}=0$$

or in this way

$$Cdu+CCx^{\frac{-c}{a}}(a+bx)^{\frac{c}{a}-\frac{e}{b}}uudx+x^{\frac{c}{a}}(a+bx)^{\frac{e}{b}-\frac{c}{a}-1}(f+gx)\frac{dx}{xx}=0.$$

§4 But if in complete generality one puts $v=Pu+X$ and sets

$$\frac{dP}{P}+\frac{(c+ex)dx}{x(a+bx)}+2Xdx=0.$$

our differential equation takes this form:

$$Pdu + PPuudx + dX + XXdx + \frac{(c+ex)Xdx}{x(a+bx)} + \frac{(f+gx)dx}{xx(a+bx)} = 0,$$

in which either P or Q can be taken arbitrarily, from which the other one will then be defined. If, for example, one takes $P = \alpha x^n$, it will be

$$X = \frac{-n}{2x} - \frac{c+ex}{2x(a+bx)} = \frac{-na-c-(nb+e)x}{2x(a+bx)}.$$

From these forms it is possible to select cases, in which the equation becomes integrable, which we will be able to see more easily from the propounded equation itself.

§5 So if we wish to look for the cases in which the propounded equation admits an integration, into which certainly the whole effort seems to have to be invested, as long as it is not possible to carry out the integration, at first certainly the form

$$z = Ax^m(a+bx)^n$$

offers itself immediately; for it to satisfy one has to define the relation between the constants a, b, c, e, f, g. Since therefore

$$\frac{dz}{z} = \frac{mdx}{x} + \frac{nbdx}{a+bx} \quad \text{and} \quad \frac{ddz}{z} - \frac{dz^2}{zz} = \frac{-mdx^2}{xx} - \frac{nbbdx^2}{(a+bx)^2},$$

it will be

$$\frac{ddz}{z} = \frac{m(m-1)dx^2}{xx} + \frac{2mnbdx^2}{x(a+bx)} + \frac{n(n-1)bbdx^2}{(a+bx)^2},$$

and hence after division by dx^2 this equation emerges:

$$\left.\begin{aligned} &\frac{m(m-1)(a+bx)}{xx} + \frac{2mnb}{x} + \frac{n(n-1)bb}{a+bx} \\ &+\frac{f+gx}{xx} + \frac{m(c+ex)}{xx} + \frac{nb(c+ex)}{x(a+bx)} \end{aligned}\right\} = 0,$$

for which to hold the two last terms collected into one

$$\frac{nb((n-1)bx+c+ex)}{x(a+bx)}$$

must admit the denominator $a+bx$ such that from this

$$a:c=b:(n-1)b+e \quad \text{or} \quad n-1=\frac{c}{a}-\frac{e}{b}$$

and

$$n=1+\frac{c}{a}-\frac{e}{b}$$

such that one has this equation:

$$\frac{m(m-1)a+mc+f}{xx}+\frac{m(m-1)b+2mnb+me+g+nbc:a}{x}=0,$$

from where, by writing the value just found instead of n, these two equations arise:

$$m(m-1)a+mc+f=0$$

and

$$m(m+1)b+\frac{(2m+1)bc}{a}-me-\frac{ce}{a}+\frac{bcc}{aa}+g=0.$$

Multiply the last equation by a and the first by $-b$, the sum will become

$$2mab+mbc-mae+bc-ce+\frac{bcc}{a}+ag-bf=0,$$

and hence

$$m=\frac{abf-aag-abc+ace-bcc}{2aab+abc-aae}$$

and

$$(m-1)a+c=\frac{abf-aag-2aab+aae}{2ab+bc-ae},$$

which values substituted in the first give

$$(bf-ag)^2+f(2abb+3bbc-3abe-bce+aee)+c(2b-e)(ab-ae+bc)+g(2aab-aae+abc-ace+bcc)=0,$$

which in resolved from yields

$$ag = bf - \frac{1}{2}(2ab - ae + bc) + \frac{c}{2a}(ae - bc) \pm (2ab - ae + bc)\sqrt{\frac{(a-c)^2}{4aa} - \frac{f}{a}}$$

or

$$g = \frac{c(ae - bc) + 2abf - (2ab - ae + bc)(a \pm \sqrt{(a-c)^2 - 4aa})}{2aa}.$$

So if the letter g has this value, our equation will have the integral

$$z = Ax^m(a + bx)^n,$$

while

$$m = \frac{a - c \pm \sqrt{(a-c)^2 - 4af}}{2a} \quad \text{and} \quad n = 1 + \frac{c}{a} - \frac{e}{b}.$$

§6 One finds integrable cases in another way, if the value of z is expanded into a series; if this series terminated, it exhibits a finite expression for z. Thus, assume

$$z = Ax^n + Bx^{n+1} + Cx^{n+2} + Dx^{n+3} + Ex^{n+4} + \text{etc.}$$

and after the substitution we will obtain:

$$\left.\begin{array}{rrrrrrr}
n(n-1)Aax^{n-2} & + & (n+1)nBax^{n-1} & + & (n+2)(n+1)Cax^n & + & (n+3)(n+2)Dax^{n+1}\text{etc.} \\
 & + & n(n-1)Ab & + & (n+1)nBb & + & (n+2)(n+1)Cb \\
+nAc & + & (n+1)Bc & + & (n+2)Cc & + & (n+3)Dc \\
 & + & nAe & + & (n+1)Be & + & (n+2)Ce \\
+Af & + & Bf & + & Cf & + & Df \\
 & + & Ag & + & Bg & + & Cg
\end{array}\right\} = 0,$$

which single terms must be reduced to zero. Thus, first it will be

$$n(n-1)a + nc + f = 0,$$

and hence

$$n = \frac{a - c \pm \sqrt{(a-c)^2 - 4af}}{2a},$$

but further

$$B = \frac{-n(n-1)b - ne - g}{(n+1)na + (n+1)c + f}A = \frac{-n(n-1)b - ne - g}{2na + c}A$$

$$C = \frac{-(n+1)nb - (n+1)e - g}{(n+2)(n+1)a + (n+2)c + f}B = \frac{-(n+1)nb - (n+1)e - g}{2((2n+1)a + c)}B$$

$$D = \frac{-(n+2)(n+1)b - (n+2)e - g}{(n+3)(n+2)a + (n+3)c + f}C = \frac{-(n+2)(n+1)b - (n+2)e - g}{3((2n+2)a + c)}C$$

etc.

Thus, this series terminates at a certain point, if having taken any positive integer number for i, zero included, it was

$$g = -(n+i)(n+i-1)b - (n+i)e.$$

But since

$$n + i = \frac{(2i-1)a - c \pm \sqrt{(a-c)^2 - 4af}}{2a}$$

and

$$n + i - 1 = \frac{(2i+1) - c \pm \sqrt{(a-c)^2 - 4af}}{2a},$$

it will be

$$g = \frac{-((2i+1)a - c \pm \sqrt{(a-c)^2 - 4af})((2i-1)ab - bc + 2ae \pm b\sqrt{(a-c)^2 - 4af})}{4af}$$

and by expanding

$$g = \frac{2abf + c(ae - bc) - a(2iiab + (2i+1)(ae - bc)) \mp (2iab + ae - bc)\sqrt{(a-c)^2 - 4af}}{2aa};$$

thus, if it was $i = -1$, which value cannot be taken here, the preceding case would emerge. Hence innumerable other similar cases are found.

§7 We can even assume the series, in which the exponents of x decrease, this way

$$z = Ax^n + Bx^{n-1} + Cx^{n-2} + Dx^{n-3} + Ex^{n-4} + \text{etc.},$$

having substituted which series our equation becomes

$$0 = \begin{cases} & & & n(n-1)Aax^{n-2} & & (n-1)(n-2)Bax^{n-3} & + & (n-2)(n-3)Cax^{n-4} + \text{etc.} \\ + & n(n-1)Abx^{n-1} & + & (n-1)(n-2)Bb & + & (n-2)(n-3)Cb & + & (n-3)(n-4)Db \\ & & + & nAc & + & (n-1)Bc & + & (n-2)Cc \\ + & nAe & + & (n-1)Be & + & (n-2)Ce & + & (n-3)De \\ & & + & Af & + & Bf & + & Cf \\ + & Ag & + & Bg & + & Cg & + & Dg \end{cases}$$

and hence it has to be

$$n(n-1)b + ne + g = 0$$

or

$$n = \frac{b - e \pm \sqrt{(b-e)^2 - 4bg}}{2b},$$

or even

$$g = -nnb + nb - ne.$$

Furthermore:

$$B = \frac{n(n-1)a + nc + f}{(2n-2)b + e}A, \quad C = \frac{(n-1)(n-2)a + (n-1)c + f}{2((2n-3)b + e)}B,$$

$$D = \frac{(n-2)(n-3)a + (n-2)c + f}{3((2n-4)b + e)}C, \quad E = \frac{(n-3)(n-4)a + (n-3)c + f}{4((2n-5)b + e)}D$$

etc.

As before, let i be a positive integer number, zero not included, and an integral will be obtained, as often as it was

$$(n-1)(n-i-1)a+(n-i)c+f=0,$$

whence

$$n=\frac{(2i+1)a-c+\sqrt{(a-c)^2-4af}}{2a}$$

as we found $g=-n((n-1)b+e)$, and hence

$$g=\frac{-((2i+1)a-c+\sqrt{(a-c)^2-4af})((2i-1)ab-bc+2ae+b\sqrt{(a-c)^2-4af})}{4aa},$$

which, as before, in expanded form yields

$$g=\frac{2abf+c(ae-bc)-a(2iiab+(2i+1)(ae-bc))-(2iab+ae-bc)\sqrt{(a-c)^2-4af}}{2aa}$$

such that hence the same cases as before arise, and even the same integrals, just written in reverse order, are obtained.

§8 But before we investigate the integral using series, our equation can be transformed into another of the same form by putting

$$z=(a+bx)^m v,$$

whence

$$\frac{dz}{z}=\frac{dv}{v}+\frac{mbdx}{a+bx}$$

and

$$\frac{ddz}{z}=\frac{ddv}{v}-\frac{mbbdx^2}{(a+bx)^2}+\frac{2mbdxdv}{v(a+bx)}+\frac{mmbbdx^2}{(a+bx)^2},$$

and after the substitution

$$\left.\begin{aligned}\frac{(a+bx)ddv}{v}+\quad\frac{2mbdxdv}{v}\quad&+\frac{m(m-1)bbdx^2}{a+bx}\\+\frac{(c+ex)dxdv}{xv}&+\frac{mb(c+ex)dx^2}{x(a+bx)}\\&+\frac{(f+gx)dx^2}{xx}\end{aligned}\right\}=0$$

let $(m-1)bx+c+ex$ become divisible by $a+bx$, and it will be

$$(m-1)b+e=\frac{bc}{a}\quad\text{and}\quad m=1+\frac{c}{a}-\frac{e}{b},$$

and our equation

$$(a+bx)ddv\left(c+\left(\frac{2bc}{a}+2b-e\right)x\right)\frac{dxdv}{x}+\left(f+\left(g+\frac{bc}{a}+\frac{bcc}{aa}-\frac{ce}{a}\right)x\right)\frac{vdx^2}{xx}=0.$$

For the sake of brevity, put

$$\frac{2bc}{a}+2b-e=\varepsilon\quad\text{and}\quad g+\frac{bc}{a}+\frac{bcc}{aa}-\frac{ce}{a}=\eta$$

such that one has this formula similar to the one propounded

$$(a+bx)ddv+(c+\varepsilon x)\frac{dxdv}{x}+(f+\eta x)\frac{vdx^2}{xx}=0,$$

which therefore is integrable, if

$$\eta=\frac{2abf+c(a\varepsilon-bc)-a(2iiab+(2i+1)(a\varepsilon-bc))\mp(2iab+a\varepsilon-bc)\sqrt{(a-c)^2-4af}}{2aa};$$

but

$$a\varepsilon-bc=2ab-ae+bc,$$

whence one has

$$\eta=\frac{2abf+c(2ab-ae+bc)-a(2(i+1)^2ab-(2i+1)(ae-bc))\mp(2(i+1)ab-ae+bc)\sqrt{(a-c)^2-4af}}{2aa}$$

$$=g+\frac{c(ab-ae+bc)}{aa},\quad\text{and hence}$$

$$g = \frac{2abf + c(ae-bc) - a(2(i+1)^2 ab - (2i+1)(ae-bc)) \mp (2(i+1)ab - ae + bc)\sqrt{(a-c)^2 - 4af}}{2aa},$$

which expression agrees with the preceding, if there one sets $-i-1$ instead of i. Therefore, it is already possible here to take so positive as negative integer for i.

§9 But it can happen that the cases which are integrable by means of the first series can also be integrated by means of the second and so two integrals can exhibited for the same equation. For, let us put that the number i for the second form exceeds the integer number i of the preceding form by the excess $\alpha - 1$ such that we write $i + \alpha - 1$ for i here. Having done this, such that both values of g are congruent, necessarily

$$2(i+\alpha)^2 ab - (2i + 2\alpha - 1)(ae - bc) = 2iiab + (2i+1)(ae-bc)$$

and

$$2(i+\alpha)ab - ae + bc = 2iab + ae - bc,$$

from which

$$\alpha ab = ae - bc$$

follows. But writing αab instead of $ae - bc$ in the first, dividing by ab

$$2(i+\alpha)^2 - 2\alpha i - 2\alpha\alpha + \alpha = 2ii + 2\alpha i + \alpha$$

results; since this holds for all values of i, we will have

$$\alpha = \frac{ae - bc}{ab},$$

which expression must be an integer number.

§10 Since we therefore found infinitely many values for the letter g, in which the propounded equation admits an integration, and one can even assign a satisfying algebraic formula for z, it is worth the effort that we consider these cases more accurately. Thus, while i denotes an arbitrary integer either positive or negative number, the expansion first done in § 7 requires these two conditions:

$$n(n-1)b + ne + g = 0$$

and

$$(n-i)(n-i-1)a + (n-i)c + f = 0,$$

from which one deduces

$$n = \frac{b - e + \sqrt{(b-e)^2 - 4bg}}{2b}$$

and

$$n - i = \frac{a - c + \sqrt{(a-c)^2 - 4af}}{2a},$$

whence

$$i = \frac{bc - ae + a\sqrt{(b-e)^2 - 4bg} - b\sqrt{(a-c)^2 - 4af}}{2ab}.$$

Thus, as often as this formula

$$\frac{bc - ae + a\sqrt{(b-e)^2 - 4bg} - b\sqrt{(a-c)^2 - 4af}}{2ab},$$

where the irrational parts can be taken both, positively or negatively, becomes equal to an either positive or negative integer number, so often the propounded equation admits an integration.

§11 If the coefficients a, b, c, e, f, g are rational, that this can happen, either each of both square roots must be rational or they must cancel each other. For the latter case,

$$aa(b-e)^2 - 4aabg = bb(a-c)^2 - 4abbf$$

or

$$4ab(ag - bf) = (ae - bc)(ae + bc - 2ab),$$

but then necessarily

$$i = \frac{bc - ae}{2ab}.$$

For the first case on the other hand, if we set

$$\sqrt{(a-c)^2-4af}=h \quad \text{and} \quad \sqrt{(b-e)^2-4bg}=k,$$

it will be

$$f=\frac{(a-c)^2-hh}{4a} \quad \text{and} \quad g=\frac{(b-e)^2-kk}{4b}.$$

Thus, if the letters f and g have such values, see whether this expression

$$\frac{bc-ae+ak-bh}{2ab}$$

is an integer number. For, if then it is a positive integer number, one will be able to exhibit the value of z by means of the first series, but if it was negative, by means of the second series. And if additionally

$$\frac{ae-bc}{ab}$$

was an integer number, the integration can be carried out in each of both ways, whence one will obtain an algebraic complete integral.

§12 One can even discover integrable cases by finding a factor multiplied by which the equation becomes integrable. For this aim, let us consider an equation of this form:

$$ddz+Qdxdz+Rzdx^2=0$$

such that

$$Q=\frac{c+ex}{x(a+bx)} \quad \text{and} \quad R=\frac{f+gx}{xx(a+bx)}.$$

And let the multiplier be

$$2pdz+qzdx$$

and hence the integrable equation

$$2pdzddz+qzdxddz+2pQdxdz^2+Qqzdx^2dz+2pRzdx^2dz+qRzzdx^3=0.$$

Set the integral equation

$$pdz^2 + qzdxdz + zzdx^2 \int Rqdx = Adx^2;$$

after subtraction of its differential from the integrable equation it has to be

$$\left.\begin{array}{r} 2Qpdxdz^2 - dpdz^2 - qdxdz^2 \\ + Qqzdx^2dz + 2Rpzdx^2dz - zdxdqdz - 2zdx^2dz \int Rqdx \end{array}\right\} = 0,$$

from where these two equations arise:

$$dp + qdx = 2Qpdx \quad \text{or} \quad \frac{dp}{p} + \frac{qdx}{p} = 2Qdx$$

and

$$Qqdx + 2Rpdx - dq - 2dx \int Rqdx = 0.$$

Put

$$\int Rqdx = S, \quad \text{it will be} \quad Rdx = \frac{dS}{q}$$

and

$$Qqdx + \frac{2PdS}{q} - dq - 2Sdx = 0 \quad \text{or} \quad dS - \frac{Sqdx}{p} = \frac{qdq}{2p} - \frac{Qqqdx}{2p}$$

and, having written the above for Qdx,

$$dS - \frac{Sqdx}{p} = \frac{qdq}{2p} - \frac{qqdp}{4pp} - \frac{q^3dx}{4pp}.$$

Let $\varkappa$ be the number whose logarithm is $= 1$, and by integration one finds

$$\varkappa^{-\int \frac{qdx}{p}} S = \int \varkappa^{-\int \frac{qdx}{p}} \left(\frac{qdq}{2p} - \frac{qqdp}{4pp} - \frac{q^3dx}{4pp} \right)$$

or

$$\varkappa^{-\int \frac{qdx}{p}} S = \frac{1}{4}C + \varkappa^{-\int \frac{qdx}{p}} \frac{qq}{4p},$$

whence

$$S = \frac{1}{4} C \varkappa^{\int \frac{qdx}{p}} + \frac{qq}{4p} = \int Rqdx,$$

and hence further

$$R = \frac{C \varkappa^{\int \frac{qdx}{p}}}{4p} + \frac{2pdq - qdp}{4ppdx} \quad \text{and} \quad Q = \frac{dp}{2pdx} + \frac{q}{2p}.$$

From these we conclude, having propounded the equation:

$$ddz + \frac{(dp + qdx)dz}{2p} + (C \varkappa^{\int \frac{qdx}{2p}} pdx + 2pdq - qdp) \frac{zdx}{4pp} = 0,$$

if it is multiplied by $2pdz + qzdx$, that the integral will be

$$pdz^2 + qzdxdz + (C \varkappa^{\int \frac{qdx}{p}} p + qq) \frac{zzdx^2}{4p} = Adx^2.$$

§13 Applying this to the case at hand, first we obtain

$$\frac{(c + ex)dx}{x(a + bx)} = \frac{dp}{2p} + \frac{qdx}{2p},$$

whence we calculate

$$\int \frac{qdx}{p} = -\log p + \frac{2c}{a} \log x + \frac{2(ae - bc)}{ab} \log(a + bx),$$

and hence

$$\varkappa^{\int \frac{qdx}{p}} = \frac{x^{\frac{2c}{a}} (a + bx)^{\frac{2(ae-bc)}{ab}}}{p}$$

and from this

$$\frac{f + gx}{xx(a + bx)} = \frac{Cx^{\frac{2c}{a}} (a + bx)^{\frac{2(ae-bc)}{ab}} dx + 2pdq - qdp}{4ppdx}.$$

But

$$q = \frac{2p(c + ex)}{x(a + bx)} - \frac{dp}{dx}$$

and thus

$$dq = \frac{2(c+ex)dp}{x(a+bx)} - \frac{2pdx(ac+2bcx+bexx)}{xx(a+bx)^2} - \frac{ddp}{dx},$$

having substituted which the equation to be resolved will be

$$\frac{4(f+gx)ppdx}{xx(a+bx)}$$

$$= Cx^{\frac{2c}{a}}(a+bx)^{\frac{2(ae-bc)}{ab}}dx + \frac{2(c+ex)pdp}{x(a+bx)} - \frac{4ppdx(ac+2bcx+bexx)}{xx(a+bx)^2} - \frac{2pddp}{dx} + \frac{dp^2}{dx}.$$

§14 But this way we get entangled in greater difficulties than if we wanted to resolve the propounded equation itself. Thus, let us take a more particular route, and let us find the conditions for the coefficients A, B, C such that this equation:

$$Ax^{\lambda}ddz + Bx^{\lambda-1}dxdz + Cx^{\lambda-2}zdx^2 = 0,$$

if multiplied by $2xdz + \alpha zdx$, becomes integrable. Thus, since the product is

$$\left.\begin{array}{ll} 2Ax^{\lambda+1}dzddz + 2Bx^{\lambda}dxdz^2 & + \alpha Bx^{\lambda-1}zdx^2dz + \alpha Cx^{\lambda-2}zzdx^3 \\ + \alpha Ax^{\lambda}zdxddz & +2Cx^{\lambda-1}zdx^2dz \end{array}\right\} = 0,$$

the integral by necessity is

$$Ax^{\lambda+1}dz^2 + \alpha Ax^{\lambda}zdxdz + \frac{\alpha}{\lambda-1}Cx^{\lambda-1}zzdx^2 = Edx^2;$$

if its differential is subtracted from it, this equation will result:

$$x^{\lambda}dxdz^2(2B-(\lambda+1)A-\alpha A) + x^{\lambda-1}zdx^2dz\left(\alpha B + 2C - \alpha\lambda A - \frac{2\alpha C}{\lambda-1}\right) = 0.$$

Hence, having annihilated each of the two terms separately, first

$$B = \frac{\alpha+\lambda+1}{2}A$$

and hence

$$\frac{\alpha(\alpha-\lambda+1)}{2}A = \frac{2(\alpha-\lambda+1)}{\lambda-1}C,$$

and thus,

$$\frac{\alpha(\alpha-\lambda+1)}{2}A=\frac{2(\alpha-\lambda+1)}{\lambda-1}C,$$

from which one finds in two ways:

$$\text{either}\quad \lambda=\alpha+1 \quad \text{or} \quad C=\frac{\alpha(\lambda-1)}{4}A.$$

Thus, two equations arise

$$\begin{array}{ll} \text{the one} & Ax^{\alpha+1}ddz+(\alpha+1)Ax^{\alpha}dxdz+Cx^{\alpha-1}zdx^2=0, \\ \text{the other} & Ax^{\lambda}ddz+\frac{1}{2}(\alpha+\lambda+1)Ax^{\lambda-1}dxdz+\frac{1}{4}\alpha(\lambda-1)Ax^{\lambda-2}zdx^2=0, \end{array}$$

each of both, if multiplied by $2xdz+\alpha zdx$, becomes integrable; for, the integral of the first will be

$$Ax^{\alpha+2}dz^2+\alpha Ax^{\alpha+1}zdxdz+Cx^{\alpha}zzdx^2=Edx^2,$$

of the second on the other hand

$$Ax^{\lambda+1}dz^2+\alpha Ax^{\lambda}dxdz+\frac{1}{4}\alpha\alpha Ax^{\lambda-1}zzdx^2=Edx^2.$$

§15 Therefore, the sum of these two equations will be rendered integrable by the same factor. Namely, this equation:

$$(Ax^{\alpha+1}Dx^{\lambda})dzz+((\alpha+1)Ax^{\alpha}+\frac{1}{2}(\alpha+\lambda+1)Dx^{\lambda-1}dxdz$$
$$+(Cx^{\alpha-1}+\frac{1}{4}\alpha(\lambda-1)Dx^{\lambda-2})zdx^2=0,$$

if multiplied by $2xdz+\alpha zdx$, yields the integral

$$(Ax^{\alpha+2}+Dx^{\lambda+1})dz^2+\alpha(Ax^{\alpha+1}+Dx^{\lambda})zdxdz+(Cx^{\alpha}+\frac{1}{4}\alpha\alpha Dx^{\lambda-1})zzdx^2=Edx^2,$$

which can be represented this way:

$$(Ax^{\alpha}+Dx^{\lambda-1})(xdz+\frac{1}{2}\alpha zdx)^2=dx^2((\frac{1}{4}\alpha\alpha A-C)x^{\alpha}zz+E)$$

such that

$$xdz + \frac{1}{2}\alpha zdx = \frac{1}{2}dx\sqrt{\frac{4E + (\alpha\alpha A - 4C)x^{\alpha}zz}{Ax^{\alpha} + Dx^{\lambda-1}}}.$$

Put

$$x^{\alpha}zz = vv, \quad \text{it will be} \quad x^{\alpha-1}z(2xdz + \alpha zdx) = 2vdv,$$

and hence

$$2xdz + \alpha zdx = \frac{2vdv}{x^{\alpha-1}z} = \frac{2dv}{x^{\frac{1}{2}\alpha-1}};$$

hence

$$2x^{1-\frac{1}{2}\alpha}dv = dx\sqrt{\frac{4E + (\alpha\alpha A - 4C)vv}{Ax^{\alpha} + Dx^{\lambda-1}}}$$

or

$$\frac{2dv}{\sqrt{4E + (\alpha\alpha A - 4C)vv}} = \frac{x^{\frac{1}{2}\alpha-1}dx}{\sqrt{Ax^{\alpha} + Dx^{\lambda-1}}}.$$

§16 To bring this equation to our form, which can be done in two ways, let us first put $\lambda = \alpha$ such that after division by x^{α} one has this equation:

$$(D + Ax)ddz + ((\alpha + \frac{1}{2})D + (\alpha + 1)Ax)\frac{dxdz}{x} + (\frac{1}{4}\alpha(\alpha - 1)D + Cx)\frac{zdx^2}{xx} = 0,$$

which multiplied by $x^{\alpha}(2xdz + \alpha zdx)$ becomes integrable, while the integral having put

$$x^{\alpha}zz = vv \quad \text{or} \quad z = x^{-\frac{1}{2}\alpha}v,$$

becomes

$$\frac{2dv}{\sqrt{4E + (\alpha\alpha A - 4C)vv)}} = \frac{x^{\frac{1}{2}\alpha-1}dx}{\sqrt{x^{\alpha-1}(D + Ax}} = \frac{dx}{\sqrt{x(D + Ax)}}.$$

Now let

$$D = a, \quad A = b, \quad \left(\alpha + \frac{1}{2}\right) a = c \quad \text{or} \quad \alpha = \frac{c}{a} - \frac{1}{2} \quad \text{and} \quad C = g,$$

and this equation will arise:

$$(a + bx)ddz + \left(c + \frac{b(a + 2c)}{2a}x\right)\frac{dxdz}{x} + \left(\frac{(2c - a)(2c - 3a)}{16a} + gx\right)\frac{zdx^2}{xx} = 0$$

such that for the propounded form

$$e = \frac{b(a + 2c)}{2a} \quad \text{and} \quad f = \frac{(2c - a)(2c - 3a)}{16a},$$

and the integral of this equation, having put

$$z = x^{\frac{1}{4} - \frac{c}{2a}} v,$$

will be

$$\frac{2dv}{\sqrt{4E + \left(\frac{b(2c-a)^2}{4aa} - 4g\right) vv}} = \frac{dx}{\sqrt{x(a + bx)}}.$$

§17 Secondly, set $\lambda = \alpha + 2$ such that after division by $x^{\alpha+1}$ this equation arises:

$$(A + Dx)ddz + \left((\alpha + 1)A + (\alpha + \frac{3}{2})Dx\right)\frac{dxdz}{x} + \left(C + \frac{1}{4}\alpha(\alpha + 1)Dx\right)\frac{zdx^2}{xx} = 0,$$

which multiplied by

$$x^{\alpha+1}(2xdz + \alpha zdx),$$

having put

$$x^{\alpha} zz = vv \quad \text{or} \quad z = x^{-\frac{1}{2}\alpha} v,$$

will have the integral

$$\frac{2dv}{\sqrt{4E+(\alpha\alpha A-4C)vv}}=\frac{x^{\frac{1}{2}\alpha-1}dx}{\sqrt{x^{\alpha}(A+Dx)}}=\frac{dx}{x\sqrt{A+Dx}}.$$

Now let

$$A=a, \quad D=b, \quad (\alpha+1)a=c \quad \text{or} \quad \alpha=\frac{c}{a}-1 \quad \text{and} \quad C=f$$

such that this equation is obtained:

$$(a+bx)ddz+\left(c+\frac{b(a+2c)}{2a}x\right)\frac{dxdz}{x}+\left(f+\frac{bc(c-a)}{4aa}x\right)\frac{zdx^2}{xx}=0,$$

and for the propounded form

$$e=\frac{b(a+2c)}{2a} \quad \text{and} \quad g=\frac{bc(c-a)}{4aa},$$

whose integral, having put

$$z=x^{\frac{1}{2}-\frac{c}{2a}}v,$$

is

$$\frac{2dv}{\sqrt{4E+\left(\frac{(c-a)^2}{a}-4f\right)vv}}=\frac{dx}{x\sqrt{a+bx}}.$$

§18 But not only as often as the propounded equation:

$$(a+bx)ddz+(c+ex)\frac{dxdz}{x}+(f+gx)\frac{zdx^2}{xx}=0$$

is contained in the one of these forms, which happens, if it was

$$\text{either} \quad e=\frac{b(c+2a)}{2a} \quad \text{and} \quad f=\frac{(2c-a)(2c-3a)}{16a}$$

$$\text{or} \quad e=\frac{b(a+2c)}{2a} \quad \text{and} \quad g=\frac{bc(c-a)}{16a},$$

it admits an integration, but also as often as, after a transformation, it is contained in the other of the two. But the transformation, as we saw above in § 8, happens by the substitution

$$z = (a+bx)^{1+\frac{c}{a}-\frac{c}{b}}v,$$

from which

$$(a+bx)ddv + (c+\varepsilon x)\frac{dxdv}{x} + (f+\eta x)\frac{vdx^2}{xx} = 0$$

arises, while

$$\varepsilon = \frac{2b(a+c)}{a} - e \quad \text{and} \quad \eta = g - \frac{ce}{a} + \frac{bc(a+c)}{aa}.$$

This equation by putting $v = x^n s$, because of

$$\frac{dv}{v} = \frac{ndx}{x} + \frac{ds}{s} \quad \text{and} \quad \frac{ddv}{v} = \frac{n(n-1)dx^2}{xx} + \frac{2ndxds}{xs} + \frac{dds}{s},$$

is transformed into this one:

$$\left.\begin{array}{rrr} (a+bx)\dfrac{dds}{s} + 2n(a+bx)\dfrac{dxds}{xs} & + n(n-1)(a+bx)\dfrac{dx^2}{xx} \\ + \quad (c+\varepsilon x)\dfrac{dxds}{xs} & + \quad n(c+\varepsilon x)\dfrac{dx^2}{xx} \\ & + \quad (f+\eta x)\dfrac{dx^2}{xx} \end{array}\right\} = 0,$$

whence these two integrable cases are found.

The first, if

$$D = a, \quad A = b, \quad \left(\alpha + \frac{1}{2}\right)a = 2na + c,$$

$$(\alpha+1)b = 2nb + \frac{2b(a+c)}{a} - e,$$

$$\frac{1}{4}\alpha(\alpha-1)a = n(n-1)a + nc + f,$$

$$C = n(n-1)b + \frac{2nb(a+c)}{a} - ne + g - \frac{ce}{a} + \frac{bc(a+c)}{aa},$$

and hence

$$\alpha = 2n - \frac{1}{2} + \frac{c}{a} \quad \text{and} \quad e = \frac{3}{2}b + \frac{bc}{a} = \frac{b(3a+2c)}{2a}$$

and

$$\left(n - \frac{1}{4} + \frac{c}{2a}\right)\left(n - \frac{3}{4} + \frac{c}{2a}\right)a = n(n-1)a + nc + f,$$

whence

$$f = \frac{cc}{4a} - \frac{c}{2} + \frac{3a}{16} = \frac{(2c-a)(2c-3a)}{16a}.$$

The other case is contained in these conditions:

$$A = a, \quad D = b, \quad (\alpha+1)a = 2na + c, \quad \left(\alpha + \frac{3}{2}\right)b = 2nb + \frac{2b(a+c)}{a} - e,$$

$$C = n(n-1)a + nc + f,$$

$$\frac{1}{4}\alpha(\alpha+1)b = n(n-1)b + \frac{2nb(a+c)}{a} - ne + g - \frac{ce}{a} + \frac{bc(a+c)}{aa},$$

whence

$$\alpha = 2n - 1 + \frac{c}{a}, \quad e = \frac{b(3a+2c)}{2a}$$

and

$$g = \frac{bc}{4a} + \frac{bcc}{4aa} = \frac{bc(a+c)}{4aa},$$

where it is clear that the number n does not contribute.

§19 Therefore, hence we obtained four integrable cases, which are:

$$
\begin{array}{lll}
1) & e = \dfrac{b(a+2c)}{2a}, & f = \dfrac{(2c-a)(2c-3a)}{16a} \\[2ex]
2) & e = \dfrac{b(a+2c)}{2a}, & g = \dfrac{bc(c-a)}{4aa} \\[2ex]
3) & e = \dfrac{b(3a+2c)}{2a}, & f = \dfrac{(2c-a)(2c-3a)}{16a} \\[2ex]
4) & e = \dfrac{b(3a+2c)}{2a}, & g = \dfrac{bc(a+c)}{4aa},
\end{array}
$$

for which we even exhibited the complete integral. Therefore, let us see how these cases apply to the condition we deduced above [§ 10] from the series, whether they are contained in it or not.

Thus, for the first we have

$$bc - ae = \frac{-ab}{2}, \quad b - e = \frac{b(a-2c)}{2a}$$

and

$$\sqrt{(a-c)^2 - 4af} = \pm \frac{a}{2},$$

whence this formula:

$$-\frac{1}{4} \mp \frac{1}{4} + \frac{\sqrt{bb(a-2c)^2 - 16aabg}}{4ab}$$

would have to be an integer number.

For the second

$$bc - ae = \frac{-ab}{2} \quad \text{and} \quad \sqrt{(b-e)^2 - 4bg} = \pm \frac{b}{2},$$

thus, this formula:

$$-\frac{1}{4} \pm \frac{1}{4} - \frac{(a-c)^2 - 4af}{2a}$$

would have to be an integer number.

For the third

$$bc - ae = \frac{-3ab}{2}, \quad b - e = \frac{-b(a+2c)}{2a}$$

and

$$\sqrt{(a-c)^2 - 4af} = \pm\frac{a}{2},$$

whence this formula:

$$-\frac{3}{4} \mp \frac{1}{4} + \frac{bb(a+2c)^2 - 16aabg}{4ab}$$

would have to be an integer number.

For the fourth

$$bc - ae = \frac{-3ab}{2} \quad \text{and} \quad \sqrt{(b-e)^2 - 4bg} = \pm\frac{b}{2},$$

whence for this formula:

$$-\frac{3}{4} \mp \frac{1}{4} - \frac{\sqrt{(a-c)^2 - 4af}}{2a}$$

would have to be an integer number.
From this it is seen that these four cases are not contained in the above condition, and hence I found completely new cases of integrability.

§20 Therefore, since these cases, in which we found the complete integral, are completely different from those for which we exhibited the particular integral above, it will be helpful to have shown how even in these cases the complete cases can be obtained; it seems that this can be done most easily as follows:

If the equation

$$Pddz + Qdxdz + Rzdx^2 = 0$$

is satisfied by the value $z = V$ such that

$$PddV + QdxdV + RVdx^2 = 0,$$

that equation will be rendered integrable if multiplied by

$$\frac{V}{P(Vdz - zdV)}.$$

For, having put

$$\int \frac{PVddz + QVdxdz + RVzdx^2}{P(Vdz - zdV)} = Sdx,$$

it will be

$$Sdx - \log P(Vdz - zdV) = \int \frac{QVdxdz + RVzdx^2 - VdPdz + PzddV + zdVdP}{P(Vdz - zdV)}$$

$$= \int \frac{Qdx}{P} - \log P + \int \frac{z(PddV + QdxdV + RVdx^2)}{P(Vdz - zdV)}.$$

But

$$PddV + QdxdV + RVdx^2 = 0,$$

and hence one has

$$Sdx = \log(Vdz - zdV) + \int \frac{Qdx}{P} + \text{Const.} = \text{Const.},$$

whence it will be

$$Vdz - zdV = C\varkappa^{-\int \frac{Qdx}{P}}dx$$

and

$$z = CV \int \frac{dx}{VV}\varkappa^{-\int \frac{Qdx}{P}},$$

which is the complete integral found from the particular one $z = V$.

§21 Since cases of both kinds are found from the propounded equation, if it is rendered integrable multiplied by the form $2pdz + qzdx$, having put $p = uu$ such that [§ 12]

$$q = \frac{2uu(c + ex)}{x(a + bx)} - \frac{2udu}{dx},$$

the integral equation will be

$$uudz^2 + qzdxdz + (C\varkappa^{\int \frac{qdx}{uu}} uu + qq)\frac{zzdx^2}{4uu} = Adx^2,$$

where

$$\varkappa^{\int \frac{qdx}{uu}} = x^{\frac{2c}{a}} \frac{(a+bx)^{\frac{2(ae-bc)}{ab}}}{uu}$$

and hence

$$\left(udz + \frac{qzdx}{2u}\right)^2 = Adx^2 - \frac{Cx^{\frac{2c}{a}}(a+bx)^{\frac{2(ae-be)}{ab}}zzdx^2}{4uu},$$

but the quantity u has to be found from this equation:

$$\frac{ddu}{dx} - \frac{(c+ex)du}{x(a+bx)} + \frac{(f+gx)udx}{xx(a+bx)} + \frac{(ac+2bcx+bexx)udx}{xx(a+bx)^2} = \frac{Cx^{\frac{2c}{a}}(a+bx)^{\frac{2(ae-bc)}{ab}}dx}{4u^2},$$

and the cases of the first kind are derived from this for $C = 0$. But this equation, having put

$$u = x^{\frac{c}{a}}(a+bx)^{\frac{ae-bc}{ab}}v,$$

goes over into this one:

$$\frac{Cx^{\frac{-2c}{a}}(a+bx)^{\frac{2(bc-ae)}{ab}}dx^2}{4v^2} = ddv + \frac{(c+ex)dxdv}{x(a+bx)} + \frac{(f+gx)vdx^2}{xx(a+bx)},$$

whose application is easier, whence for $C = 0$ the quantity v has to satisfy this equation:

$$(a+bx)ddv + \frac{(c+ex)dxdv}{x} + \frac{(f+gx)vdx^2}{xx} = 0$$

such that the complete value is obtained from the particular value. But if we put

$$u = x^m(a+bx)^n,$$

it will be

$$\frac{1}{4}Cx^{\frac{2c}{a}-4m}(a+bx)^{\frac{2(ae-bc)}{ab}-4n}$$

$$=\frac{m(m-1)}{xx}+\frac{(f-mc+(g-me+2mnb)x)}{xx(a+bx)}+\frac{ac+(2-n)bcx+(n-1)(nbb-be)xx}{xx(a+bx)^2},$$

and hence so the exponents m and n together with the constant C as the relation between the coefficients a, b, c, e, f, g have to be defined from this equation:

$$\frac{1}{4}Cx^{\frac{2c}{a}-4m+2}(a+bx)^{\frac{2(ae-bc)}{ab}-4n+2}=m(m-1)(a+bx)^2$$

$$+(a+bx)(f-mc+(g-me+2mnb)x)+ac+(2-n)bcx+(n-1)(nb-e)bxx.$$

§22 From this many cases result that we want to expand:
First case. If the the exponent is

$$\frac{2(ae-bc)}{ab}-4n+2=2 \quad \text{or} \quad n=\frac{ae-bc}{2ab},$$

in which is has to be

$$\frac{2c}{a}-4m+2=0 \quad \text{or} \quad m=\frac{a+c}{2a}$$

such that one has

$$\frac{1}{4}C(a+bx)^2=\frac{cc-aa}{4aa}(a+bx)^2+(f-\frac{c(a+c)}{2a}+\left(g-\frac{bc(a+c)}{2aa}\right)x)(a+bx)$$

$$+ac+(2-n)bcx+(n-1)(nb-e)bxx,$$

where the last term has to be divisible by $a+bx$, which can be happen in two ways.

1) Either it is $n=1$; and hence

$$e=\frac{2ab+bc}{a}=\frac{b(2a+c)}{a},$$

and so it will be

$$\frac{1}{4}C(a+bx) = \frac{cc-aa}{4aa}(a+bx) + f - \frac{c(a+c)}{2a} + (g - \frac{bc(a+c)}{2aa})x + c,$$

whence

$$\frac{1}{4}Ca = \frac{cc-aa}{4a} + f + \frac{c(a-c)}{2a} = f - \frac{(a-c)^2}{4a}$$

and

$$\frac{1}{4}Cb = \frac{b(cc-aa)}{4aa} + g - \frac{bc(a+c)}{2aa} = g - \frac{b(a+c)^2}{4aa}.$$

Therefore,

$$bf - ag - \frac{b(a-c)^2}{4a} + \frac{b(a+c)^2}{4a} = 0 \quad \text{or} \quad g = \frac{bf}{a} + \frac{bc}{a} = \frac{b(c+f)}{a}$$

and

$$\frac{q}{2u} = \frac{u(c+ex)}{x(a+bx)} - \frac{du}{dx} = x^{\frac{c-a}{2a}}(c+ex) - \frac{a+c}{2a}x^{\frac{c-a}{2a}}(a+bx) - bx^{\frac{a+c}{2a}}$$

or

$$\frac{q}{2u} = \frac{a(c-a)+b(a+c)x}{2a}x^{\frac{c-a}{2a}}.$$

As a logical consequence the integral equation reads

$$\left(x^{\frac{a+c}{2a}}(a+bx)dz + \frac{a(c-a)+b(a+c)x}{2a}x^{\frac{c-a}{2a}}zdx\right)^2 = Adx^2 - \left(\frac{f}{a} - \frac{(a-c)^2}{4aa}\right)x^{\frac{c-a}{a}}(a+bx)^2zzdx^2.$$

2) Or it is

$$n = \frac{ae-bc}{ab} = \frac{ae-bc}{2ab},$$

and hence

$$e = \frac{bc}{a} \quad \text{and} \quad n = 0;$$

hence

$$\frac{1}{4}C(a+bx) = \frac{cc-aa}{4aa}(a+bx) + f - \frac{c(a+c)}{2a} + \left(g - \frac{bc(a+c)}{2aa}\right)x + c + \frac{bc}{a}x,$$

thus,

$$\frac{1}{4}Ca = \frac{cc-aa}{4a} + f + \frac{c(a-c)}{2a} = f - \frac{(a-c)^2}{4a}$$

and

$$\frac{1}{4}Cb = \frac{b(cc-aa)}{4aa} + g + \frac{bc(a-c)}{2aa} = g - \frac{b(a-c)^2}{4aa},$$

whence one concludes

$$bf = ag \quad \text{or} \quad g = \frac{bf}{a};$$

this is the case, in which the propounded equation is divisible by $a+bx$, and so has no difficulty.

§23 *The second* case is the one in which

$$\frac{2(ae-bc)}{ab} - 4n + 2 = 1 \quad \text{and} \quad \frac{2c}{a} - 4m + 2 = 0$$

and hence

$$m = \frac{a+c}{2a} \quad \text{and} \quad n = \frac{ab+2ae-2bc}{4ab}$$

such that we have

$$\frac{1}{4}C(a+bx) = \frac{cc-aa}{4aa}(a+bx)^2 + (f - \frac{c(a+c)}{2a} + (g + \frac{b(a+c)(a-2c)}{4aa}x)(a+bx)$$
$$+ac + (2-n)bcx + (n-1)(nb-e)bxx,$$

which cases are again split into two subcases:

1) Either it is $n=1$ and hence

$$2ae - 2bc = 3ab \quad \text{and} \quad e = \frac{3ab + 2bc}{2a} = \frac{b(3a + 2c)}{2a},$$

whence

$$\frac{1}{4}C = \frac{cc - aa}{4aa}(a + bx) + f + \frac{c(a - c)}{2a} + gx + \frac{b(a + c)(a - 2c)}{4aa}x$$

and thus

$$\frac{1}{4}C = \frac{cc - aa}{4a} + f + \frac{c(a - c)}{2a} = f - \frac{(a - c)^2}{4a}$$

and

$$0 = \frac{b(cc - aa)}{4aa} + g + \frac{b(a + c)(a - 2c)}{4aa} \quad \text{or} \quad g = \frac{bc(a + c)}{4aa},$$

which is the fourth case in § 19.

2) Or it is

$$n = \frac{ae - bc}{ab} = \frac{ab + 2ae - 2bc}{4ab}$$

and hence

$$e = \frac{b(a + 2c)}{2a} \quad \text{and} \quad n = \frac{1}{2};$$

thus, the equation, if divided by $a + bx$, becomes

$$\frac{1}{4}C = \frac{cc - aa}{4aa}(a + bx) + f + \frac{c(a - c)}{2a} + (g + \frac{b(a + c)(a - 2c)}{4aa})x + \frac{bc}{2a}x$$

such that

$$\frac{1}{4}C = \frac{cc - aa}{4a} + f + \frac{c(a - c)}{2a} = f - \frac{(a - c)^2}{4a}$$

and

$$\frac{b(cc - aa)}{4aa} + \frac{bc}{2a} + \frac{b(a + c)(a - 2c)}{4aa} + g = 0 \quad \text{or} \quad g = \frac{bc(c - a)}{4aa},$$

which was case 2) in § 19.

§24 *The third* case is that one in which

$$\frac{2(ae-bc)}{ab}-4n+2=1 \quad \text{and} \quad \frac{2c}{a}-4m+2=1$$

and hence

$$m=\frac{a+2c}{4a} \quad \text{and} \quad n=\frac{ab+2ae-2bc}{4ab},$$

and so we will have

$$\frac{1}{4}Cx(a+bx)=\frac{(a+2c)(2c-3a)}{16aa}(a+bx)^2+(f-\frac{c(a+2c)}{4a}+(g+\frac{b(a+2c)(a-2c)}{8aa})x)(a+bx)$$
$$+ac+(2-n)bcx+(n-1)(nb-e)bxx,$$

whose last term is rendered divisible by $a+bx$ in two ways.

1) If $n=1=\frac{ab+2ae-2bc}{4ab}$ and hence $e=\frac{b(3a+2c)}{2a}$, whence

$$\frac{1}{4}Cx=\frac{(a+2c)(2c-3a)}{16aa}(a+bx)+f+\frac{c(3a-2c)}{4a}+gx+\frac{b(a+2c)(a-2c)}{8aa}x$$

such that it has to be

$$\frac{(a+2c)(2c-3a)}{16a}+f-\frac{c(2c-3a)}{4a}=0 \quad \text{or} \quad f=\frac{(2c-a)(2c-3a)}{16a}$$

and

$$\frac{1}{4}C=\frac{b(a+2c)(2c-3a)}{16aa}+g+\frac{b(a+2c)(a-2c)}{8aa}=g-\frac{b(a+2c)^2}{16aa},$$

which was case 3) in § 19.

2) If $n=\frac{ae-bc}{ab}=\frac{ab+2ae-2bc}{4ab}$ or $e=\frac{b(a+2c)}{2a}$ and $n=\frac{1}{2}$, hence

$$\frac{1}{4}Cx=\frac{(a+2c)(2c-3a)}{16aa}(a+bx)+f-\frac{c(a+2c)}{4a}+gx+\frac{b(a+2c)(a-2c)}{8aa}x+c+\frac{bc}{2a}x,$$

and hence

$$f+\frac{(a+2c)(2c-3a)}{16a}+\frac{c(3a-2c)}{4a}=0 \quad \text{or} \quad f=\frac{(2c-a)(2c-3a)}{16a}$$

and

$$\frac{1}{4}C=\frac{b(a+2c)(2c-3a)}{16aa}+g+\frac{b(a+2c)(a-2c)}{8aa}+\frac{bc}{2a}=g-\frac{b(a-2c)^2}{16aa},$$

which was case 1) in § 19.

§25 *The fourth* case is that one in which

$$\frac{2(ae-bc)}{ab}-4n+2=0 \quad \text{and} \quad \frac{2c}{a}-4m+2=0$$

and hence

$$m=\frac{a+c}{2a} \quad \text{and} \quad n=\frac{ab+ae-bc}{2ab}$$

such that we have

$$\frac{1}{4}C=\frac{cc-aa}{4aa}(a+bx)^2+(a+bx)(f-\frac{c(a+c)}{2a}+(g+\frac{b(aa-cc)}{2aa})x)$$
$$+ac+\frac{3ab-ae+bc}{2a}cx-\frac{(ab-ae+bc)(ab-ae-bc)}{4aa}xx,$$

whence cancelling each power separately we conclude

$$\frac{bb(cc-aa)}{4aa}+bg+\frac{bb(aa-cc)}{2aa}-\frac{(ab-ae+bc)(ab-ae-bc)}{4aa}=0,$$
$$\frac{b(cc-aa)}{2a}+bf-\frac{bc(a+c)}{2a}+ag+\frac{b(aa-cc)}{2a}+\frac{c(3ab-ae+bc)}{2a}=0,$$

from the last

$$g=\frac{e(e-2b)}{4b},$$

from the first on the other hand

$$bf + ag = \frac{c(e-2b)}{2}$$

and hence

$$f = \frac{(e-2b)(2bc-ae)}{4bb},$$

which are the two conditions; but then one will have to take

$$\frac{1}{4}C = \frac{cc-aa}{4} + af - \frac{c(a+c)}{2} + ac = af - \frac{1}{4}(a-c)^2.$$

§26 *The fifth* case is that one in which

$$\frac{2(ae-bc)}{ab} - 4n + 2 = 0 \quad \text{and} \quad \frac{2c}{a} - 4m + 2 = 1$$

and hence

$$m = \frac{2c+a}{4a} \quad \text{and} \quad n = \frac{ab+ae-bc}{2ab}$$

such that we have

$$\frac{1}{4}Cx = \frac{(2c+a)(2c-3a)}{16aa}(a+bx)^2 + (a+bx)(f - \frac{c(2c+a)}{4a} + (g + \frac{b(a-c)(2c+a)}{4aa})x)$$
$$+ac + \frac{3ab-ae+bc}{2a}cx - \frac{(ab-ae+bc)(ab-ae-bc)}{4aa}xx$$

and hence

$$\frac{bb(2c+a)(2c-3a)}{16aa} + bg + \frac{bb(a-c)(2c+a)}{4aa} - \frac{(ab-ae+bc)(ab-ae-bc)}{4aa} = 0$$

or

$$g = \frac{(b-2e)(3b-2e)}{16b},$$

$$\frac{(2c+a)(2c-3a)}{16} + af - \frac{c(2c+a)}{4} + ac = 0 \quad \text{or} \quad f = \frac{(2c-a)(2c-3a)}{16a}$$

and

$$\frac{1}{4}C = \frac{(ab-ae+bc)^2}{4ab}.$$

§27 *The sixth* case is that one in which

$$\frac{2(ae-bc)}{ab}-4n+2=0 \quad \text{and} \quad \frac{2c}{a}-4m+2=2$$

and hence

$$m=\frac{c}{2a} \quad \text{and} \quad n=\frac{ab+ae-b}{2ab}$$

such that we have

$$\frac{1}{4}Cxx=\frac{c(c-2a)}{4aa}(a+bx)^2+(a+bx)^2(f-\frac{cc}{2a}+(g+\frac{bc(a-c)}{2aa})x)$$
$$+ac+\frac{3ab-ae+bc}{2a}cx-\frac{(ab-ae+bc)(ab-ae-bc)}{4aa}xx,$$

whence it has to be:

$$\frac{c(c-2a)}{4}+af-\frac{cc}{2}+ac=0 \quad \text{or} \quad f=\frac{c(c-2a)}{4a},$$
$$\frac{bc(c-2a)}{2a}+bf-\frac{bcc}{2a}+ag+\frac{bc(a-c)}{2a}+\frac{3ab-ae+bc}{2a}c=0$$

or

$$g=\frac{-c(2ab-2ae+bc)}{4aa}$$

and

$$\frac{1}{4}C=\frac{-bc(2ab-2ae+bc)}{4aa}-\frac{(b-e)^2}{4}=bg-\frac{1}{4}(b-e)^2.$$

§28 But since for these cases $u=x^m(a+bx)^n$, it will be

$$\frac{q}{2u}=x^{m-1}(a+bx)^{n-1}(c+ex)-mx^{m-1}(a+bx)^n-nbx^m(a+bx)^{n-1}$$

or

$$\frac{q}{2u}=x^{m-1}(a+bx)^{n-1}(c-ma+(e-(m+n)b)x),$$

whence one calculates the integral equation

$$x^{2m}(a+bx)^{2n}(dz+\frac{c-ma+(e-(m+n)b)x}{x(a+bx)}zdx)^2$$
$$=Adx^2-\frac{1}{4}Cx^{\frac{2c}{a}-2m}(a+bx)^{\frac{2(ae-bc)}{ab}-2n}zzdx^2$$

or it will be

$$dz+\frac{c-ma+(e-(m+n)b)x}{x(a+bx)}zdx=\frac{dx\sqrt{A-\frac{1}{4}Cx^{\frac{2c}{a}-2m}(a+bx)^{\frac{2(ae-bc)}{ab}-2n}zz}}{x^m(a+bx)^n}.$$

Therefore, for the cases that we found the integrals of the propounded equation

$$(a+bx)ddz+(c+ex)\frac{dxdz}{x}+\frac{(f+gx)zdx^2}{xx}=0$$

will look as follows.

CASE 1

$$m=\frac{a+c}{2a},\quad n=1,\quad e=\frac{b(2a+c)}{a},\quad g=\frac{b(c+f)}{a}\quad \text{and}\quad \frac{1}{4}C=\frac{f}{a}-\frac{(a-c)^2}{4aa}.$$

Therefore, the integral will be

$$dz+\frac{a(c-a)+b(a+c)x}{2ax(a+bx)}zdx=dx\sqrt{\frac{A}{x^{\frac{a+c}{a}}(a+bx)^2}-\frac{4af-(a-c)^2}{4aaxx}zz}.$$

CASE 2

$$e=\frac{bc}{a},\quad g=\frac{bf}{a},\quad m=\frac{a+c}{2a},\quad n=0\quad \text{and}\quad \frac{1}{4}C=\frac{4af-(a-c)^2}{4aa}.$$

Therefore, the integral will be

$$dz+\frac{a(c-a)+b(c-a)x}{2ax(a+bx)}zdx=dx\sqrt{\frac{A}{x^{\frac{a+c}{a}}}-\frac{4af-(a-c)^2}{4aaxx}zz}$$

or

$$dz+\frac{(c-a)zdx}{2ax}=dx\sqrt{\frac{A}{x^{\frac{a+c}{a}}}+\frac{(a-c)^2-4af}{4aaxx}zz}.$$

Case 3

$$e=\frac{b(3a+2c)}{2a},\quad g=\frac{bc(a+c)}{4aa},\quad m=\frac{a+c}{2a},\quad n=1\quad\text{and}\quad \frac{1}{4}C=\frac{4af-(a-c)^2}{4a},$$

whence the integral is

$$dz+\frac{a(c-a)+bcx}{2ax(a+bx)}zdx=dx\sqrt{\frac{A}{x^{\frac{a+c}{a}}(a+bx)^2}+\frac{((a-c)^2-4af)zz}{4axx(a+bx)}}.$$

Case 4

$$e=\frac{b(a+2c)}{2a},\quad g=\frac{bc(c-a)}{4aa},\quad m=\frac{a+c}{2a},\quad n=\frac{1}{2}\quad\text{and}\quad \frac{1}{4}C=\frac{4af-(a-c)^2}{4a};$$

thus, the integral is

$$dz+\frac{(c-a)zdx}{2ax}=dx\sqrt{\frac{A}{x^{\frac{a+c}{a}}(a+bx)}+\frac{((a-c)^2-4af)zz}{4axx(a+bx)}}.$$

Case 5

$$e=\frac{b(3a+2c)}{2a},\quad f=\frac{(2c-a)(2c-3a)}{16a},\quad m=\frac{a+2c}{4a},\quad n=1\quad\text{and}\quad \frac{1}{4}C=g-\frac{b(a+2c)^2}{16aa},$$

whence the integral will be

$$dz+\frac{a(2c-a)+b(2c+a)x}{4ax(a+bx)}zdx=dx\sqrt{\frac{A}{x^{\frac{a+2c}{a}}(a+bx)^2}+\frac{(b(a+2c)^2-16aag)zz}{16aax(a+bx)}}.$$

Case 6

$$e=\frac{b(a+2c)}{2a},\quad f=\frac{(2c-a)(2c-3a)}{16a},\quad m=\frac{a+2c}{4a},\quad n=\frac{1}{2}\quad\text{and}\quad \frac{1}{4}C=g-\frac{b(a-2c)^2}{16aa},$$

whence the integral will be

$$dz+\frac{(2c-a)zdx}{4ax}=dx\sqrt{\frac{A}{x^{\frac{a+2c}{a}}(a+bx)}+\frac{(b(a+2c)^2-16aag)zz}{16aax(a+bx)}}.$$

CASE 7

$$f = \frac{(e-2b)(2bc-ae)}{4bb}, \quad g = \frac{e(e-2b)}{4b}, \quad m = \frac{a+c}{2a}, \quad n = \frac{ab+ae-bc}{2ab} \quad \text{and} \quad \frac{1}{4}C = af - \frac{1}{4}(a-c)^2,$$

whence the integral will be

$$dz + \frac{c-a+(e-2b)x}{2x(a+bx)}zdx = dx\sqrt{\frac{A}{x^{2m}(a+bx)^{2n}} + \frac{((a-c)^2-4af)zz}{4xx(a+bx)^2}}.$$

CASE 8

$$f = \frac{(2c-a)(2c-3a)}{16a}, \quad g = \frac{(b-2e)(3b-2e)}{16b}, \quad m = \frac{2c+a}{4a}, \quad n = \frac{ab+ae-bc}{2ab} \quad \text{and} \quad \frac{1}{4}C = \frac{(ab-ae+bc)^2}{4ab};$$

thus, the integral will be

$$dz + \frac{2c-a+(2e-3b)x}{4x(a+bx)}zdx = dx\sqrt{\frac{A}{x^{2m}(a+bx)^{2n}} - \frac{(ab-ae+bc)^2zz}{4abx(a+bx)^2}}.$$

CASE 9

$$f = \frac{c(c-2a)}{4a}, \quad g = \frac{-c(2ab-2ae+bc)}{4aa}, \quad m = \frac{c}{2a}, \quad n = \frac{ab+ae-bc}{2ab} \quad \text{and} \quad \frac{1}{4}C = bg - \frac{1}{4}(b-e)^2,$$

whence the integral is

$$dz + \frac{c-(e-b)x}{2x(a+bx)}zdx = dx\sqrt{\frac{A}{x^{2m}(a+bx)^{2n}} + \frac{((b-e)^2-4bg)zz}{4(a+bx)^2}}.$$

§29 Aside from these nine cases, in which two relations among the coefficients are prescribed, initially we found innumerable integrable cases in two ways. In the first (§ 6), the algebraic integrals of this form:

$$z = Ax^n + Bx^{n+1} + Cx^{n+2} + Dx^{n+3} + \text{etc.}$$

can be assigned, if, while i denotes an arbitrary positive number, it was

$$n(n-1)a + nc + f = 0$$

and

$$(n+i)(n+i-1)b + (n+i)e + g = 0.$$

In the other (§ 8) the integral is of this form:

$$z = (a+bx)^{\frac{ab-ae+bc}{ab}}(Ax^n + Bx^{n+1} + Cx^{n+2} + \text{etc.}),$$

if it was

$$n(n-1)a + nc + f = 0$$

and

$$(n+i)(n+i-1)b + (n+i)\left(\frac{2bc}{a} + 2b - e\right) + g + \frac{bc}{a} + \frac{bcc}{aa} - \frac{ce}{a} = 0.$$

Even if these are just particular integrals, from them the complete integrals are nevertheless easily determined.